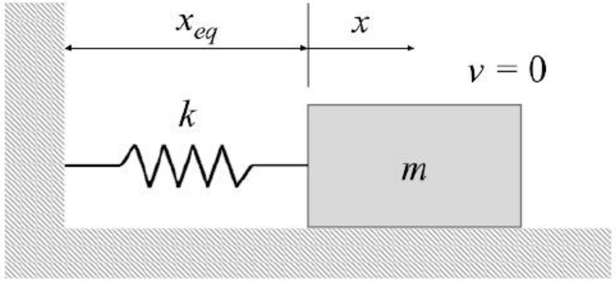
SIMPLE HARMONIC MOTION

Periodic Motion

- Periodic motion: Any motion of a system that is continuously and identically repeated. E.g.: simple pendulum, object moving in circular path with constant speed, simple harmonic motion of spring-mass system etc.
- Time period: It is the time taken to complete one cycle of periodic motion.

Simple Harmonic Motion

- It is a type of periodic motion where force exerted on the oscillating object is proportional to the displacement. E.g.: Spring-mass system.
- For the spring-mass system,



$$F = -kx \Rightarrow$$

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad [\omega^2 = \frac{k}{m}]$$

The solution to this D.E is,

$$\boldsymbol{x = Asin\omega t + Bcos\omega t}$$

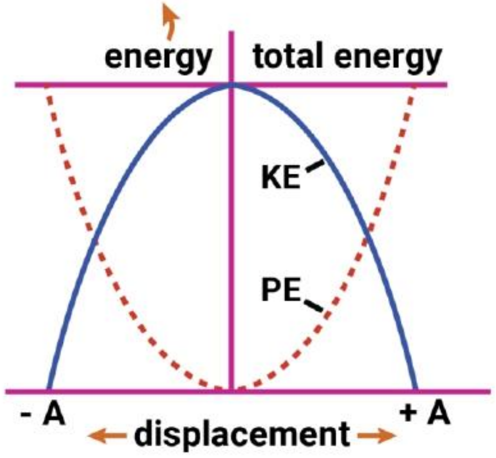
- Applying initial conditions: x(t=0)=a and $v(t=0)=\dot{x}(t=0)=0 \Rightarrow$

$$a = A\sin(\omega 0) + B\cos(\omega 0) = B \text{ and}$$

$$0 = A\omega\cos(\omega t) - B\omega\sin(\omega t)\Big|_{t=0} = A\omega \Rightarrow A = 0$$

$$\therefore, \text{the solution is,}$$

$$x(t) = a\cos(\omega t)$$



Thus the velocity, $v(t) = \dot{x}(t) = -a\omega\sin(\omega t)$

- The kinetic energy can be found as,

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ma^2\omega^2\sin^2\omega t$$

- The potential energy of the spring for the displacement x is,

$$U = \text{Workdone to displace the spring by distance } x$$

$$= -\int F.dx = -\int -kx\,dx$$

$$\therefore U = \frac{1}{2}kx^2 = \frac{1}{2}ka^2\cos^2\omega t$$

Thus total energy, $E = T + U = \frac{1}{2}ma^2\omega^2\sin^2\omega t + \frac{1}{2}ka^2\cos^2\omega t$

$$E = \frac{1}{2}ka^2\sin^2\omega t + \frac{1}{2}ka^2\cos^2\omega t = \frac{1}{2}ka^2(\cos^2\omega t + \sin^2\omega t) = \boldsymbol{\frac{1}{2}ka^2}$$

$$\Rightarrow \text{Energy is constant}$$

Velocity and Acceleration of SHM

- We already got the solution of the SHM, with specific initial conditions($At\ t = 0, x = a\ and\ v = 0$) as,

$$\boldsymbol{x(t) = a\cos(\omega t)}$$

- We have also seen the velocity as,

$$\boldsymbol{v(t) = \dot{x} = -a\omega\sin(\omega t)}$$

- The Acceleration is,

$$\boldsymbol{a(t) = \dot{v} = -a\omega^2 cos\omega t}$$

- Thus position, velocity and acceleration changes sinusoidally with different phases with respect to time.
- Thus we can conclude that:

I) At x=0 (or $t = \frac{\pi}{2\omega}, \frac{3\pi}{2\omega}, \frac{5\pi}{2\omega}..$) position →

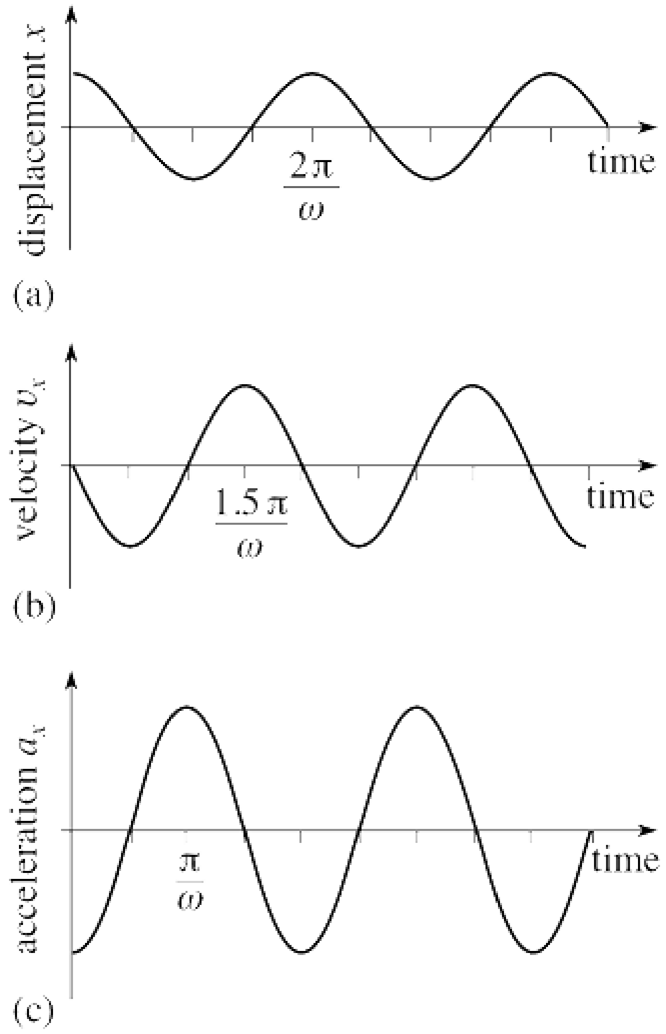
i) velocity is maximum

ii) acceleration is zero

II) At x=$\boldsymbol{a}$ (or $t = \frac{\pi}{\omega}, \frac{2\pi}{\omega}, \frac{3\pi}{\omega}..$) position →

i) velocity is zero

ii) acceleration is maximum



Simple Pendulum

- Simple pendulum under small oscillations follows SHM. The tangential force to the bob of the pendulum is,

$$F_x = -mg\sin\theta \approx -mg\theta = -mg\left(\frac{x}{l}\right)$$

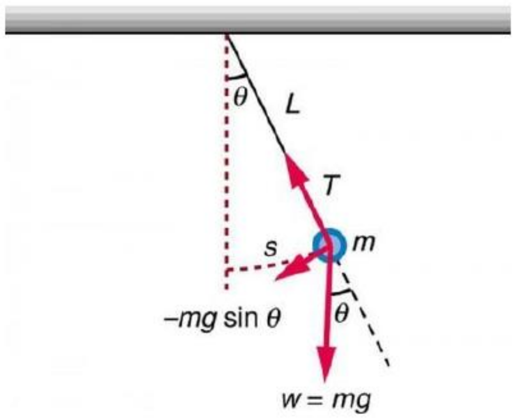
$$\Rightarrow F = -kx, \qquad where\ k = \frac{mg}{l}$$

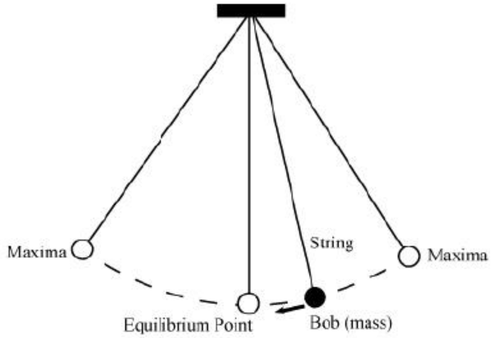
- Thus simple pendulum follows SHM with angular frequency,

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{l}}$$

Thus time period of oscillation,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{g}{l}}$$





- Here the potential energy can be found as follows:

$$U = mg\Delta\mathrm{x} = \mathrm{mg(l - lcos}\theta) = mgl(1 - cos\theta) = 2mglsin^2\left(\frac{\theta}{2}\right) - - -(1)$$

Let U=0 when x=0(or $\theta = 0$). At x=A(the maximum amplitude-> $\theta = \frac{A}{l}$) the potential energy is,

$$U_{max} = 2mglsin^2\left(\frac{\frac{A}{l}}{2}\right) = 2mglsin^2\left(\frac{A}{2l}\right) - - - - (2)$$

- Also we can find the kinetic energy as follows:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x})^2 = \frac{1}{2}m(\dot{x})^2 = \frac{1}{2}m\left(l\dot{\theta}\right)^2 = \frac{1}{2}ml^2\dot{\theta}^2 \quad - - -(3)$$

Thus,
Total energy, $E = U + T = U_{max} \Rightarrow$

$$E = 2mglsin^2\left(\frac{\theta}{2}\right) + \frac{1}{2}ml^2\dot{\theta}^2 = 2mglsin^2\left(\frac{A}{2l}\right)$$

The solution to equation motion of simple pendulum,

$$\frac{d\theta^2}{dt^2} + \frac{g}{l}\theta = 0,$$

is give by,

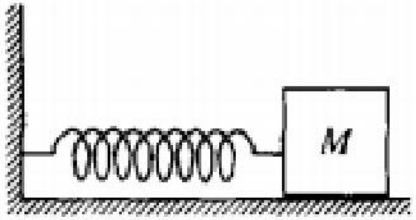
$$\theta = \theta_0\cos(\omega t + \phi)$$

Where, angular frequency, $\omega = \sqrt{\frac{g}{l}}$ and ϕ is the initial phase(the phase when t=0)

Exercises

1)

An ideal massless spring is fixed to the wall at one end, as shown. A block of mass M attached to the other end of the spring oscillates with amplitude A on a frictionless, horizontal surface. The maximum speed of the block is v_m. The force constant of the spring is



(A) $\frac{Mgv_m}{2A}$ (B) $\frac{Mv_m^2}{2A}$ (C) $\frac{Mv_m^2}{A^2}$ (D) $\frac{Mv_m^2}{2A^2}$

2)

A mass m is attached to a spring with a spring constant k. If the mass is set into simple harmonic motion by a displacement d from its equilibrium position, what would be the speed, v, of the mass when it returns to the equilibrium position?

$(A)\ v=\sqrt{\frac{md}{k}}$ $(B)\ v=\sqrt{\frac{kd}{m}}$ $(C)\ v=\sqrt{\frac{kd}{mg}}$ $(D)\ v=d\sqrt{\frac{k}{m}}$

Homework

1) Find the period and frequency of a simple pendulum 1.000 m long at a location where $g = 9.8\ m/s^2$.

2) A 1.50-kg mass on a spring has displacement as a function of time given by the equation

$$x(t) = (7.40\ \text{cm})\cos[(4.16\ \text{s}^{-1})t - 2.42]$$

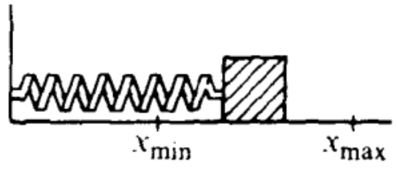
Find (a) the time for one complete vibration; (b) the force constant of the spring; (c) the maximum speed of the mass; (d) the maximum force on the mass; (e) the position, speed, and acceleration of the mass at $t = 1.00$ s; (f) the force on the mass at that time.

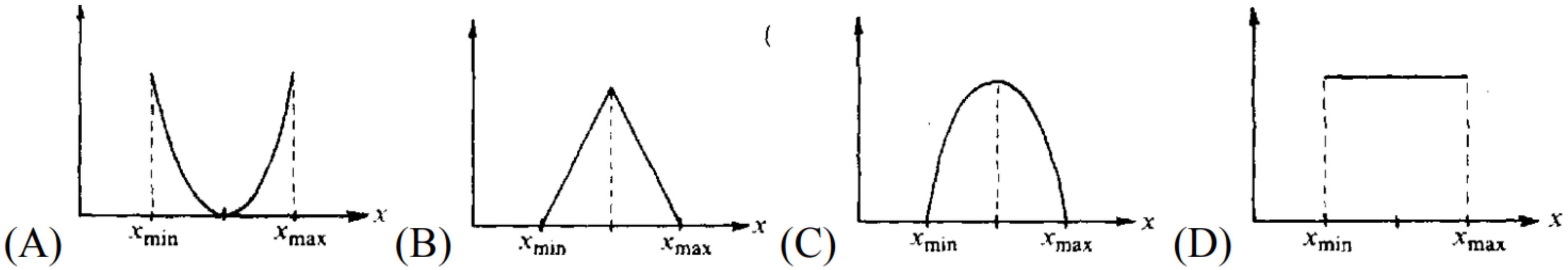
3) A 10.0-kg mass is traveling to the right with a speed of 2 m/s on a smooth horizontal surface when it collides with and sticks to a second 10 Kg mass that is initially at rest but is attached to a light spring with force constant 110 N/m (a) Find the frequency, amplitude, and period of the subsequent oscillations. (b) How long does it take the system to return the first time to the position it had immediately after the collision?

4)

An apple weighs 1.00 N. When you hang it from the end of a long spring of force constant 1.5 N/m and negligible mass, it bounces up and down in SHM. If you stop the bouncing and let the apple swing from side to side through a small angle, the frequency of this simple pendulum is half the bounce frequency. (Because the angle is small, the back-and-forth swings do not cause any appreciable change in the length of the spring.) What is the unstretched length of the spring (with the apple removed)?

5) A block oscillates without friction on the end of a spring as shown. The minimum and maximum lengths of the spring as it oscillates are, respectively, x_{min} and x_{max}. The graphs below can represent quantities associated with the oscillation as functions of the length x of the spring.



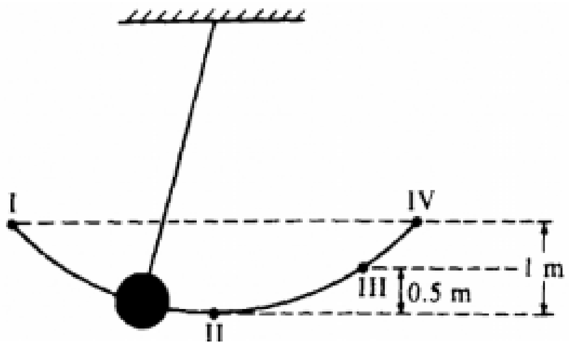


a) Which graph can represent the total mechanical energy of the block-spring system as a function of x ?

b) Which graph can represent the kinetic energy of the block as a function of x ? (A) A (B) B (C) C (D) D

6)

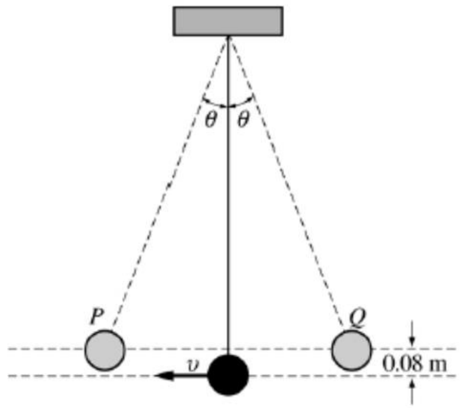
A ball swings freely back and forth in an arc from point I to point IV, as shown. Point II is the lowest point in the path, III is located 0.5 meter above II, and IV is I meter above II. Air resistance is negligible.



a) If the potential energy is zero at point II, where will the kinetic and potential energies of the ball be equal?
(A) At point II (B) At some point between II and III
(C) At point III (D) At some point between III and IV

b) The speed of the ball at point II is most nearly
(A) 3.0 m/s (B) 4.5 m/s (C) 9.8 m/s (D) 14 m/s

7) A simple pendulum consists of a bob of mass 0.085 kg attached to a string of length 1.5 m. The pendulum is raised to point Q, which is 0.08 m above its lowest position, and released so that it oscillates with small amplitude θ between the points P and Q as shown below.



(a) On the figures, draw free-body diagrams showing and labeling the forces acting on the bob in each of the situations described when bob is at lowest position and the bob is at Q.

(b) Calculate the speed v of the bob at its lowest position.

(c) Calculate the tension in the string when the bob is passing through its lowest position.